Inequality 1

12 October 2023 18:30

P = set of positive numbers in the real number system.

$x > 0$ if $x \in P$.

<u>Property</u>:- Every real number x has one and only one of the following properties:-

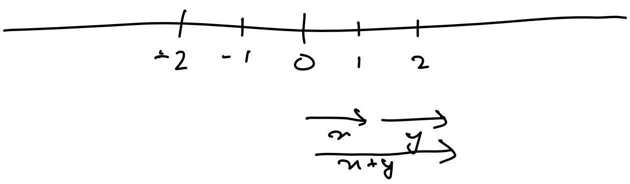
(i) $x = 0$

(ii) $x \in P$ (i.e, $x > 0$)

(iii) $-x \in P$ (i.e, $-x > 0$)

<u>Property</u>: If $x, y \in P$ then $x + y \in P$

<u>Property</u>: If $x, y \in P$ then $xy \in P$



•> If $a < b$ and c is any number then $a + c < b + c$

•> If $a < b$ and $c > 0$ then $ac < bc$

<u>Proof</u>:-

$a < b \iff b - a > 0 \iff (b+c) - (a+c) > 0 \iff b + c > a + c$

$b > a \iff (b - a) > 0 \iff (b - a)c > 0 \iff bc > ac$

Q> Prove that $a < 0,\ b < 0 \Rightarrow ab > 0$

Ans:- $-a > 0,\ -b > 0 \iff ab > 0$

Q> Prove that if $a > 1 \Rightarrow a^2 > 1$

Ans:- $a > 1 \iff (a - 1) > 0 \iff (a-1)^2 > 0 \iff a^2 - 2a + 1 > 0 \iff a^2 > 2a - 1$

$a > 1 \iff a + a > 2 \iff 2a > 2 \iff 2a - 1 > 1 \iff a^2 > 1$

Q> If $0 < a < 1$ then $a^2 < a$. Prove it.

Ans:- $a^2 > 0$, If $a > 1 \iff a^2 > a$

then $a < 1$ $\in P \cap \{a^2 > a\}^c = \{a^2 < a\} \cup \{a^2 = a\}$

$a^2 = a \Rightarrow a^2 - a > 0 \Rightarrow a(a-1) > 0 \Rightarrow a = 1$

$\Rightarrow 0 < a < 1 \Rightarrow a^2 < a$.

Absolute value of a real number $x \in \mathbb{R}$ is $|x|$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Q> $|x| \geq 0$. Prove it

Ans:- $|x| = x$ if $x > 0 \Rightarrow |x| = x$ if $x \in P \Rightarrow |x| \in P \Rightarrow |x| > 0$

$|x| = -x$ if $x < 0 \Rightarrow |x| = -x$ if $-x \in P$

$|x| = -x$ if

$|x| > 0$ when $x > 0$ or $x < 0$

If $x \notin P \Rightarrow -x \notin P \Rightarrow x = 0 \Rightarrow |x| = x = 0$

$|x| \geq 0$.

Q> Prove $|-x| = |x|$

Ans:- $$|-x| = \begin{cases} -x & \text{if } -x \geq 0 \longrightarrow x \leq 0 \\ x & \text{if } -x < 0 \longrightarrow x > 0 \end{cases}$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$|x| - |-x| = \begin{cases} x - x & \text{if } x > 0 \\ -x - (-x) & \text{if } x < 0 \\ 0 - 0 & \text{if } x = 0 \end{cases} = 0$$

$\Rightarrow |x| - |-x| = 0 \Rightarrow |x| = |-x|$

Q) $|x|^2 = x^2$.

Ans :- $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$|x|^2 = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases} = x^2$

Triangle Inequality :-

For any pair of real numbers a and b,

$$|a+b| \leq |a| + |b|$$

Moreover, equality holds iff $ab \geq 0$

Proof :- $|a+b| \geq 0$ $\quad |a+b|^2 = (a+b)^2$

$|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2 = |a|^2 + 2ab + |b|^2 \ldots [\text{as } |x|^2 = x^2]$

$\leq |a|^2 + 2|ab| + |b|^2 = (|a| + |b|)^2$

$|a+b|^2 \leq (|a| + |b|)^2$ $\qquad x^2 = |x|^2$

$\Rightarrow x^2 \geq 0$

$|a+b||a+b| \leq (|a| + |b|)(|a| + |b|)$

$\Rightarrow |a+b| \leq (|a| + |b|)$

$ab \leq |ab|$, If $ab \geq 0 \Rightarrow |ab| = ab \Rightarrow |a||b| = |ab|$

$\Rightarrow a > 0, b > 0 \Rightarrow |a| > 0, |b| > 0$

$a = |a|, b = |b|$

$|ab| = ab = |a||b|$

Equality holds for $ab \geq 0$

HomeWork :- $(a, b, x \in \mathbb{R})$

1> $|ab| = |a||b|$

2> $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ with $b \neq 0$

3> $|x| \leq -b \iff -b \leq x \leq b$

4> $\big||a| - |b|\big| \leq |a - b|$

General Form of Triangle Inequality :-

For real numbers $x_1, x_2, \ldots, x_n$ we get,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

Proof :- Idea similar to triangle inequality. (HomeWork)